

# A trace of inflation in the local behavior of cosmological constant

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Assuming the existence of a cosmological constant depending on time, we study the evolution of this field in a local region of spacetime. Solving the standard equations of Einstein Relativity in the weak field approximation we find two asymptotes in the behavior of the cosmological constant. Their meaning is the existence of an inflationary era both in the far past and in the future. A trace of the initial acceleration of the Universe can be found also in the local behavior of cosmological constant.

Keywords: cosmology – cosmological constant – inflationary universe – dark energy

## I. INTRODUCTION

The recent discovery of the accelerated expansion of the Universe, obtained studying the redshift of Supernovae [1, 2] and the Cosmic Microwave Background Radiation [3], can be understood introducing a cosmological constant  $\Lambda$  in the standard Friedman model. The resulting  $\Lambda$ CDM model is so used to describe the evolution of the Universe with three fundamental ingredients: Dark Energy due to the cosmological constant (69.2%), Cold Dark Matter and Baryonic Matter (30.8%). In the standard  $\Lambda$ CDM model, the cosmological constant does not depend on time, but the model must admit a period of inflation at the very beginning that cannot be explained with a very small and constant  $\Lambda$ . Immediately after the big bang, scalar fields or dilaton fields or "ad hoc" large values of  $\Lambda$  were introduced to obtain the inflation era. Our aim is to see what we can learn about the cosmological constant studying its local behavior. To this aim, we assume that the cosmological constant depends on time, as several authors have already done before us (see for example: Feng & Li [4], Overduin & Cooperstock [5]), and that its value today is  $\Lambda_0 \simeq 1.1 \times 10^{-52} m^{-2}$  (as estimated by Planck satellite data of the Cosmic Background Radiation [3]), even if it may have been different in a far past. Anyway, if the value of the cosmological constant is small, the effects caused by its existence on the local geometry can be treated in the weak field approximation [6].

## II. THE MODEL

Using the notations of Landau and Lifshits [7], we start from the Einstein field equations in the form

$$R_{\mu\nu} = (T_{\mu\nu})_{eff} - \frac{1}{2}g_{\mu\nu}(T)_{eff}, \quad (1)$$

and from a metric

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad (2)$$

where  $\eta_{\mu\nu} = (1, -1, -1, -1)$  is the flat Minkowski metric and we obtain the linearized Einstein field equations neglecting terms  $|h_{\mu\nu}|^2 \ll 1$

$$\square h_{\mu\nu} = -2(T_{\mu\nu})_{eff} + g_{\mu\nu}(T)_{eff}, \quad (3)$$

where  $\square = \partial_\mu \partial^\mu$  and

$$(T_{\mu\nu})_{eff} = \frac{8\pi G}{c^4} T_{\mu\nu} + \Lambda(t)g_{\mu\nu}. \quad (4)$$

The equation (3) was obtained imposing the gauge condition  $\partial_\mu \sigma^\mu_\nu = 0$  [8, 9], where

$$\sigma_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h. \quad (5)$$

We focus our attention on a finite region of space of volume  $V$  and we assume a perfect fluid energy - momentum tensor in the form

$$T_{\mu\nu} = [\rho(r) + \delta\rho(t)]u_\mu u_\nu, \quad (6)$$

where there are a central source of radius  $R$  with such an energy density  $\rho(r)$  that  $\rho(r) = 0$  for  $r > R$  and also an amount of energy density  $\delta\rho(t)$  spread all over  $V$ . Furthermore  $u^\mu \simeq (1, 0, 0, 0)$  in the weak field approximation. Hence, being  $M = \int \rho(r)d^3x$ , the Einstein equation (3), in the region  $V$  outside the source  $M$  (that is for  $r > R$ ), for the 00 component can be written:

$$\frac{1}{c^2} \frac{\partial^2 h_{00}}{\partial t^2} - \nabla^2 h_{00} = -\frac{8\pi G}{c^4} \delta\rho(t) + 2\Lambda(t)g_{00} + \frac{8\pi G}{c^4} \delta\rho(t)h_{00}. \quad (7)$$

We choose

$$g_{00} = 1 - \frac{2GM}{c^2 r} + \alpha(t)r + \beta(t)r^2 + \gamma(t) = 1 + h_{00}, \quad (8)$$

where  $\alpha(t)$ ,  $\beta(t)$  and  $\gamma(t)$  are unknown functions of time. Since

$$\nabla^2 h_{00} = \frac{d^2 h_{00}}{dr^2} + \frac{2}{r} \frac{dh_{00}}{dr}, \quad (9)$$

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from (8) we calculate

$$\frac{dh_{00}}{dr} = \frac{2GM}{c^2 r^2} + \alpha + 2\beta r, \quad (10)$$

and

$$\frac{d^2 h_{00}}{dr^2} = -\frac{4GM}{c^2 r^3} + 2\beta. \quad (11)$$

Note that from the constraint  $\partial_\mu (T^\mu_{\nu})_{eff} = 0$  we can add the equation

$$\frac{8\pi G}{c^4} \delta \dot{\rho} + \dot{\Lambda} = 0, \quad (12)$$

where a dot means derivative with respect to  $ct$ , and we have

$$\frac{8\pi G}{c^4} \delta \rho = A - \Lambda(t), \quad (13)$$

where the constant  $A$  is such that  $\delta \rho > 0$ . Then putting  $\tilde{\Lambda}(t) = A + \Lambda(t)$  and substituting (8), (9), (10), (11) and (13) in the equation (7), we obtain

$$\ddot{\alpha} r + \ddot{\beta} r^2 + \ddot{\gamma} - 6\beta - \frac{2\alpha}{r} = -4A + 3\tilde{\Lambda} - \frac{2GM\tilde{\Lambda}}{c^2 r} + \tilde{\Lambda}\alpha r + \tilde{\Lambda}\beta r^2 + \tilde{\Lambda}\gamma. \quad (14)$$

Finally we can find the unknown  $\alpha(t)$ ,  $\beta(t)$  and  $\tilde{\Lambda}(t)$  functions from the conditions:

$$\begin{cases} -6\beta = 3\tilde{\Lambda} \\ -2\alpha = -\frac{2GM\tilde{\Lambda}}{c^2} \\ \ddot{\alpha} = \tilde{\Lambda}\alpha \\ \ddot{\beta} = \tilde{\Lambda}\beta \end{cases} \Rightarrow \begin{cases} \beta = -\frac{\tilde{\Lambda}}{2} \\ \alpha = \frac{GM\tilde{\Lambda}}{c^2} \\ \frac{GM\ddot{\Lambda}}{c^2} = \frac{GM\tilde{\Lambda}^2}{c^2} \\ -\frac{\ddot{\Lambda}}{2} = -\frac{\tilde{\Lambda}^2}{2} \end{cases}. \quad (15)$$

Solving the differential equation, i.e.,

$$\ddot{\Lambda} = \tilde{\Lambda}^2 \Rightarrow \tilde{\Lambda}(t) = 6^{1/3} \wp \left[ \frac{ct + C_1}{6^{1/3}}, (0, C_2) \right], \quad (16)$$

we have derived the behavior of the cosmological constant as a function of time in terms of the Weierstrass  $\wp$  function and two integration constants:  $C_1$  and  $C_2$ . A plot of the function  $\Lambda = \tilde{\Lambda} - A$  is shown in Figure 1.

More difficult is to find  $\gamma(t)$  from the remaining equation:

$$\ddot{\gamma} = -4A + \tilde{\Lambda}\gamma, \quad (17)$$

that can be solved by a numerical integration. A particular solution of this differential equation, in the simple case of  $C_2 = 0$  (when  $\tilde{\Lambda} = 6/c^2 t^2$ ), is

$$\gamma = \frac{6A}{\tilde{\Lambda}} - \frac{N\tilde{\Lambda}}{A} = \frac{6}{1 + \Lambda/A} - N(1 + \Lambda/A) \quad (18)$$

and, considering that in our model  $A > \Lambda$ , with a suitable choice of the constant  $N$ , we obtain  $|\gamma(t)| < 1$  in a large range of time and the linear approximation works.

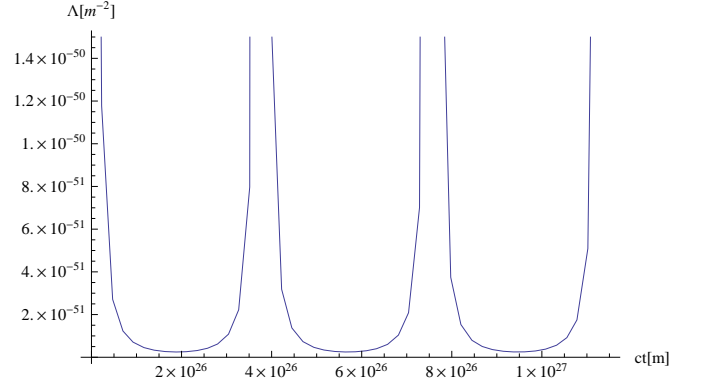


FIG. 1: The Weierstrass elliptic  $\wp$  function in the case of  $C_1 = 0$ ,  $C_2 = 10^{-155}$  and  $A = 0$ . We highlight the periodic behavior of such function.

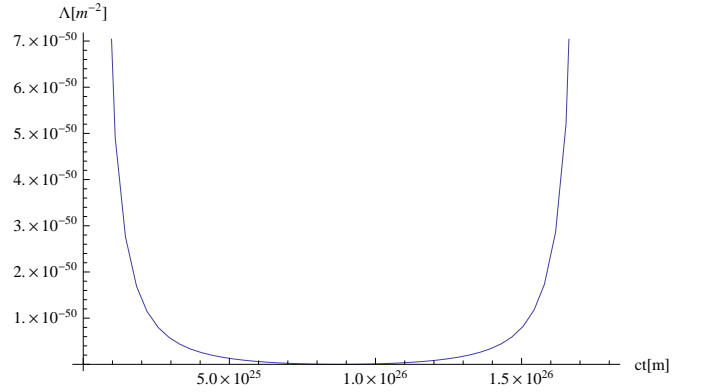


FIG. 2: An example of a possible solution for the cosmological constant  $\Lambda(t)$  fixing the constants as  $C_1 = 0$ ,  $C_2 = 10^{-153}$  and  $A \approx 10^{-51} m^{-2}$ . We highlight how, for these values of  $C_1$ ,  $C_2$  and  $A$ , the function at today time ( $t = 13.7$  billions of years, i.e.,  $ct \approx 1.3 \times 10^{26} m$ ) acquires the expected value of  $\Lambda_0 = 10^{-52} m^{-2}$ . This value is in proximity of the minimum of the function.

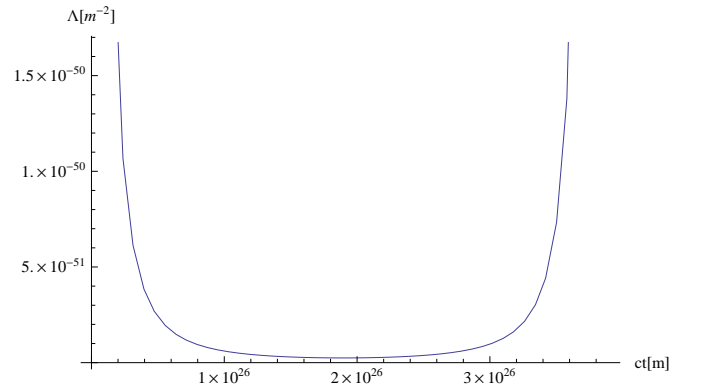


FIG. 3: The cosmological constant  $\Lambda(t)$  obtained in the limit case  $A = 0$ , fixing the constants as  $C_1 = 0$ ,  $C_2 = 10^{-155}$ . We highlight how, for these values of  $C_1$ ,  $C_2$ , the cosmological constant at today time ( $t = 13.7$  billions of years, i.e.,  $ct \approx 1.3 \times 10^{26} m$ ) acquires the expected value of  $\Lambda_0 = 10^{-52} m^{-2}$ . This value is in proximity of the minimum of the function.

### III. CONCLUSIONS

We obtained a first result determining the  $g_{00}$  component of the metric tensor (8) that is:

$$g_{00} = 1 - \frac{2GM}{c^2 r} + \frac{GM\tilde{\Lambda}(t)}{c^2} r - \frac{\tilde{\Lambda}(t)}{2} r^2 + \gamma(t), \quad (19)$$

where the fourth term induces locally a repulsive effect, the same that globally produces the accelerating expansion of the Universe. If we compare this term with the corresponding term in the Schwarzschild - de Sitter spacetime [10] we interpret the coefficient before  $r^2$  as an effective cosmological constant  $\Lambda_{eff} = 3\tilde{\Lambda}/2$ .

But the plot (Figure 1) of Weierstrass elliptic  $\wp$  - function is even more interesting. It shows a periodic function that is almost constant between two asymptotes. If the first asymptote is in  $t \simeq 0$  (it occurs fixing  $C_1 \simeq 0$ ), the rapid decrease of the cosmological constant seems to denote the post inflation phase of the Universe from which, a long era of almost constant value of  $\Lambda$  starts. The amplitude of this era is ruled by the constant  $C_2$  that can be tuned to obtain a lapse of time at least equal to the age of the universe. Finally, the constant  $A$  can be chosen in such a way that the minimum of  $\Lambda(t)$  approximatively corresponds to the today value of the cosmological constant. This way, fixing in a suitable manner the arbitrary constants  $A$ ,  $C_1$  and  $C_2$ , it is always possible to make the model in agreement with the experimental cosmological constraints: an example is shown in Figure 2. But it is

surprising that the model works also in the case of vanishing  $A$  and  $\gamma$ . We show in Figure 3 the plot of the cosmological constant obtained fixing only  $C_1$  and  $C_2$  and the cosmological constraints are satisfied as well. This case is mathematically more appealing because the number of free adjustable parameters is smaller, but it must admit the existence of negative energy density  $\delta\rho < 0$  that, anyway, is often invoked in several physical theories, such as Casimir effect, traversable wormholes, warp drives, black hole evaporation, or as a property of some kind of exotic matter.

Independently on the choice of the constants, the  $\Lambda$  function has always the two asymptotes (except the trivial case  $C_2 = 0$  with only one asymptote for  $t \rightarrow 0$ ) whose meaning is an inflationary behavior of the cosmological constant. Of course, going back into the past, when the cosmological constant becomes larger and larger, there will be an instant when the linear approximation can be no longer applied. Anyway we consider an important result to have obtained, with a so simple model and without introducing any other auxiliary fields besides  $\Lambda(t)$ , the prediction of a past and also of a future inflationary era.

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